

Generalized 2d dilaton gravity with matter fields

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Abstract

We extend the classical integrability of the CGHS model of 2d gravity [1] to a larger class of models, allowing the gravitational part of the action to depend more generally on the dilaton field and, simultaneously, adding fermion- and $U(1)$ -gauge-fields to the scalar matter. On the other hand we provide the complete solution of the *most* general dilaton-dependent 2d gravity action coupled to *chiral* fermions. The latter analysis is generalized to a chiral fermion multiplet with a non-abelian gauge symmetry as well as to the (anti-)self-dual sector $df = \pm * df$ of a scalar field f .

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1 The Setting and Our Results

One of the most influential papers written on low-dimensional gravity models was the one of Callan, Giddings, Harvey and Strominger [1]. Motivated by studying Hawking radiation, they proved the complete classical solvability of string inspired dilaton gravity [2] coupled minimally to massless scalar fields. Here we study a similar problem: First we want to see how the analysis changes, if the dilaton part of the action is replaced by a more general one. Secondly, we want to advocate the study of the coupling to fermion and gauge fields.

The most general 2d gravity action for a metric \mathbf{g} and a dilaton field Φ which yields second order differential equations is of the form [3]:

$$I_{gdil}[\mathbf{g}, \Phi] = -\frac{1}{2} \int_{\mathcal{M}} d^2x \sqrt{-\det \mathbf{g}} [U(\Phi)\mathcal{R}(\mathbf{g}) + V(\Phi)\mathbf{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + W(\Phi)]. \quad (1)$$

Here $\mathcal{R}(\mathbf{g})$ denotes the Ricci scalar of the Levi-Civita connection of \mathbf{g} and U , V , and W are some arbitrary (reasonable) functions (“potentials”) of the dilaton. Eq. (1) reduces to the original string inspired dilaton action upon the choice $U(\Phi) = \exp(-2\Phi)/\pi$, $V = -4U$, and $W = -4\lambda^2 U$.¹

Generalizing the standard actions for real massless scalar fields f and fermionic fields Ψ in curved spacetime [4] by allowing for dilaton dependent couplings $\beta(\Phi)$ and $\gamma(\Phi)$, one has:

$$I_{scal} = \frac{1}{2} \int d^2x \beta(\Phi) \sqrt{-\det \mathbf{g}} \mathbf{g}^{\mu\nu} \partial_\mu f \partial_\nu f, \quad (2)$$

$$I_{ferm}[\mathbf{e}, \Psi] = \frac{1}{2} \int_{\mathcal{M}} d^2x \gamma(\Phi) \det(\mathbf{e}_\rho^b) \left\{ i \left(\bar{\Psi} \sigma^a \mathbf{e}_a^\mu D_\mu \Psi \right) + \text{herm. conj.} \right\} \quad (3)$$

The indices μ and a take values in $0, 1$ and $+, -$, respectively. \mathbf{e}_a^μ is the inverse of \mathbf{e}_μ^a , the component-matrix of the zweibein $\mathbf{e}^\pm = \mathbf{e}_\mu^\pm dx^\mu$, related to \mathbf{g} as usual:

$$\mathbf{g} = 2\mathbf{e}^+\mathbf{e}^- \equiv \mathbf{e}^+ \otimes \mathbf{e}^- + \mathbf{e}^- \otimes \mathbf{e}^+. \quad (4)$$

D_μ denotes the covariant derivative: $D_\mu = \partial_\mu + \frac{1}{2}\varpi_\mu\sigma^3$, where ϖ_μ is the (torsion-free) spin connection. (In the presence of an additional $U(1)$ gauge

¹In the notation of [1], with the other sign convention for \mathcal{R} .

field \mathcal{A} , D_μ is understood to contain also the standard \mathcal{A}_μ part, cf. below). The basic elements of the Clifford algebra have been represented by Pauli matrices $\vec{\sigma}$, furthermore, with $\sigma^\pm \equiv \frac{1}{2}(\sigma^1 \pm i\sigma^2)$ and $\bar{\Psi} \equiv \Psi^\dagger \sigma^1$, where Ψ is a two-component complex column vector (the entries of which may be taken anti-commuting or commuting, as one prefers, since we will stay on the classical level throughout this paper).

The global $U(1)$ -symmetry of I_{ferm} may be turned into a local one by the standard procedure: $D_\mu \rightarrow D_\mu + i\mathcal{A}_\mu$, where \mathcal{A} is an (abelian) connection one-form. Dynamics for \mathcal{A} is generated by

$$I_{U(1)} = \frac{1}{4} \int_{\mathcal{M}} d^2x \sqrt{-\det \mathbf{g}} \alpha(\Phi) \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} , \quad (5)$$

where $\mathcal{F} = d\mathcal{A}$ and $\alpha(\Phi)$ is the again dilaton dependent coupling.

The results and the organization of the paper are as follows:

In [Sec. 2](#) we derive the general field equations of the action

$$I = I_{dil} + I_{scal} + I_{ferm} + I_{U(1)} . \quad (6)$$

In [Sec. 3](#) we provide the *general* (local) solution to the coupled system (6) restricted as follows:² The “potential” $W(\Phi)$ is determined by the freely chosen functions $U(\Phi)$ and $V(\Phi)$ up to the choice of two (real) constants a and b through the relation

$$W(\Phi) = \exp \left(- \int^\Phi \frac{V(z)}{U'(z)} dz \right) (4a U(\Phi) + 2b) , \quad (7)$$

²By “general solution” we always mean the following in this paper: All solutions of the field equations of the respective Lagrangian — a coupled system of partial differential equations — without imposing any boundary conditions. Thereby we identify all constants and functions parametrizing the general solution which are related by gauge transformations ($U(1)$, diffeomorphism, or Lorentz, also not restricted by any boundary or overlap conditions). Thus the solutions are local solutions, solutions on trivial topology \mathbb{R}^2 , which are also not yet geodesically complete or maximally extended in general. A maximal extension of these solutions and the study of solutions and their parameters on non-trivial topologies, as done in [5] for the case of I_{dil} alone, is not contained in the present analysis.

$\beta(\Phi)$ is constant, $\gamma(\Phi)$ arbitrary, and $\alpha(\Phi)$ is determined up to one constant by a similar equation as the one for W (Eq. (29) below), which includes $\alpha = 0$ (no gauge fields).

The treatment in Sec. 3 constitutes a reasonable generalization of the CGHS model. The latter satisfies Eq. (7) with $a = 0$ and $b = -2\lambda^2$, the coupling to the scalar fields is minimal, $\beta \equiv 1$, and in that model there are no fermions or gauge fields ($\gamma = \alpha = 0$). The class of models considered in Sec. 3 contains also e.g. the one-parameter generalization of the CGHS model considered in [6] (with the *same* a and b as above). Similarly, we also generalize the recent (independent) considerations of Cavaglià et al. [7], where they consider fermions coupled to generalized dilaton gravity; for $a = 0$ they provide the general solution to the fermion-gravity system, while for the more complicated case $a \neq 0$ they find the stationary solutions only. Some further related and in part complementary work is [8].

Note that as we allow also for $U(1)$ gauge fields, the classically solvable models considered in Sec. 3 incorporate generalizations of the massless Schwinger model to a non-trivial gravitational sector. For reasons of better interpretation, all the matter dependence of the metric is expressed in terms of the energy momentum tensor in that section, furthermore.

We do not know how to find the general solution of the field equations of the total gravity-matter action I for completely arbitrary choices of U, V, W, α, β , and γ . However, as we will show in [Sec. 4](#) and discuss briefly below, the general system (6) may be solved in the case of *chiral* fermions (with or without $U(1)$ gauge fields) and no scalar fields ($\beta = 0$). This solution may be generalized, furthermore, to the presence of non-abelian gauge fields with a chiral fermion multiplet in the fundamental representation. Also we may allow for additional torsion dependent terms in the gravitational part of the action. In Sec. 4 we thus generalize the observation of W. Kummer [9] that the Katanaev-Volovich model [10] of 2d gravity with torsion may be solved when coupled to chiral fermions.

In [Sec. 5](#) we will see that for scalar fields coupled to the gravity action

I_{gdil} there is an analog of the above chiral fermion solutions. These are the (anti-)selfdual solutions $df = \pm * df$, where the star denotes the Hodge dual (with respect to the dynamical metric). As we will show, such solutions exist for constant coupling β only. This excludes the particularly interesting case of spherically reduced 4d scalar fields, otherwise still incorporated in $I_{gdil} + I_{scal}$, since there $\beta \propto \Phi^2$ (with an appropriate definition of Φ). For a minimal coupling $\beta = const$, however, the general (anti-)self-dual solutions may be written down explicitly. More specifically, we will be able to find a mapping between the self-dual sector of $I_{gdil} + I_{scal}$ and the chiral sector $I_{gdil} + I_{ferm}$. The former will be seen to be a subsector of the latter. In a way this might be called a classical version of “bosonization”.

We finally remark that many of the results presented in this paper have already been obtained in [11] as an outcome of joint work.

Let us discuss the solutions in some further detail:

If there are no matter fields present [12, 13], the metric \mathbf{g} always has a Killing vector and thus may be brought into the form

$$\mathbf{g} = 2dx^0dx^1 + h_0(x^0)(dx^1)^2 . \quad (8)$$

For a given Lagrangian (1), there is a one-parameter family of functions h_0 , which may be written down explicitly in terms of the “potentials” U, V , and W . The dilaton field Φ , moreover, also respects the Killing symmetry, being an explicitly known function of x^0 , determined by U, V , and W . In the case of pure gravity, the space of local solutions is one-dimensional (“generalized Birkhoff theorem”).

Maximal extension of the local solutions (8) generically leads to a variety of spacetimes with black holes and kinks [5, 14]. For a given model I_{gdil} with a “generic” choice of U, V , and W , there are globally smooth solutions on two-surfaces of arbitrary non-compact topology; without matter fields and for a fixed topology of spacetime this solution space (space of global solutions to the field equations modulo gauge transformations) is still finite

dimensional.

For the local solutions found in Sec. 4 (but without gauge fields) and those found in Sec. 5, Eq. (8) is modified into

$$\mathbf{g} = 2dx^0dx^1 + [h_0(x^0) + k_0(x^0)h_1(x^1)](dx^1)^2, \quad (9)$$

where h_0 is the same function as the one in the pure gravity case, Eq. (8) above, and the dilaton field $\Phi = \Phi(x^0)$ remains unchanged, too. The respective matter field depends on the null coordinate x^1 only. The function k_0 in Eq. (9) is determined completely by U, V and W . Thus all the matter dependence in (9) may be put into the function h_1 . More explicitly we will find

$$h_1 = -2 \int^{x^1} \mathbf{T}_{11}(u) du, \quad (10)$$

where $\mathbf{T}_{11} = \mathbf{T}_{11}(x^1)$ is the only non-vanishing component of the energy momentum tensor of the respective matter field. The space of local solutions is infinite dimensional now, being parametrized by the initial data for the matter field and the one constant in h_0 . Moreover, by means of matter fields one may generate transitions between various sectors of the *pure* gravity solutions (beside the generation of completely new, not yet investigated sectors, certainly); in several instances this will also include the possibility of a black hole formation due to the presence of matter fields (but cf. also the remarks on the missing “Choptuik effect” following below).

Allowing for a non-vanishing $U(1)$ gauge field in Sec. 4 there is an additional contribution $K_0(x^0)E^2(x^1)(dx^1)^2$ to \mathbf{g} , where K_0 is again determined by means of U, V, W and E is the electric field of \mathcal{A} : $E \equiv -\alpha * dA$; up to a constant of integration, $E(x^1)$ is determined by the initial data of the fermion field. An analogous form of the solution holds for n chiral fermion generations with e.g. $SU(n)$ gauge symmetry; $E^2(x^1)$ is then just replaced by $\text{tr}(E^2)(x^1)$.

We finally remark that a “superposition” of the scenarios of Secs. 4 and 5 is possible: The general solution of the chiral fermion *and* (anti-)selfdual scalar field sector of the *general* system (6) (or any non-abelian generalization thereof) may be put together easily from the formulas of those sections.

The metric of the solutions found in Sec. 3 may be brought into the form

$$\mathbf{g} = \frac{4(2a\phi + b) dx^+ dx^-}{(1 + ax^+x^-)^2 W(U^{-1}(\phi))}, \quad (11)$$

where the function ϕ contains all the dependence on the matter fields. In terms of the total energy momentum tensor $\mathbf{T}_{\mu\nu}$ of the matter fields it has the form given in Eq. (46) below. In the case of no gauge fields ($\alpha = 0$) ϕ simplifies somewhat, since then $\mathbf{T}_{+-} = 0$ and $\mathbf{T}_{\pm\pm} = \mathbf{T}_{\pm\pm}(x^\pm)$. For the CGHS model, furthermore, in addition $W(U^{-1}(\phi)) = -4\lambda^2 \phi$, $a = 0$ and $b = -2\lambda^2$, so that the conformal factor in Eq. (11) reduces to $1/\phi$ and Eqs. (46) may be seen to reproduce the standard result $\phi = \int dx^+ \int dx^+ \mathbf{T}_{++}(x^+) + \int dx^- \int dx^- \mathbf{T}_{--}(x^-) + 2\lambda^2 x^+ x^-$.

The latter solution retains its form when fermionic matter is added to the action of the CGHS model or when fermionic matter replaces the scalar one.³ It appears to us that fermionic matter may have its advantages over scalar matter: An energy momentum shock wave, e.g., may be generated simply by a discontinuity in the phase of the fermions (as opposed to scalar fields where one needs the “square root” of a delta function). More important, there is no problem in quantizing 2d massless fermions, while quantized massless scalar fields, strictly speaking, do not exist in two dimensions (cf. [15]).

The metric (11) may be expected to describe interesting generalizations of the CGHS black holes. We will not pursue such global aspects in this paper, however.

Upon restriction to the chiral and selfdual sector of the fermion and scalar fields, respectively, the solution (11) has to be of the form (9) up to a change of coordinates. We will also not construct the necessary diffeomorphism explicitly, although this might provide some further insights into the physics of the solutions.

Despite the many physically attractive features of the CGHS model, such as the possibility to discuss Hawking radiation in a classically solvable model,

³Note that this is no more true, when we also turn on a $U(1)$ gauge field; \mathbf{T}_{++} , e.g., depends on both coordinates x^\pm then and ϕ contains also \mathbf{T}_{+-} .

there is also one qualitative difference to the spherically symmetric sector of 4d Einstein gravity with scalar fields, which may be described by $I_{gdil} + I_{scal}$ for some other specific choice of the potentials U, V, W and β : In the Einstein theory the density of matter has to surpass a threshold before a black hole can be formed by the collapse of matter; below this threshold the matter will just scatter apart again (cf., e.g., the numerical work of Choptuik [16]). In the CGHS model, on the other hand, already the smallest possible contribution to the energy momentum tensor of a scalar field will generate a black hole from the Minkowski vacuum.⁴

We did not check, if this undesirable feature is shared by all the models considered in Sec. 3. The solutions found in the subsequent Secs. 4 and 5 do miss this Choptuik effect, however, at least if we are interested in true curvature singularities when speaking of black holes.⁵ This may be seen as follows: The scalar curvature of a metric of the form (9) is

$$\mathcal{R} = -\ddot{h}_0(x^0) - h_1(x^1) \ddot{k}_0(x^0). \quad (12)$$

Suppose the matter-free solution with h_0 describes a flat or at least non-singular spacetime for some choice of the parameter in the vacuum solution. Then a curvature singularity can result only from the divergence of the second term in (12) for some x^μ . However, a smooth choice of initial data for the matter fields yields a smooth function h_1 . Thus a divergence of \mathcal{R} can result only when $\ddot{k}_0(x^0)$ blows up for some value of x^0 . As a consequence of the explicit form of h_1 , Eq. (10), there is again no threshold for a curvature singularity to form: if k_0 gives rise to such a singularity, this singularity will show up *whenever* h_1 is just non zero. We suspect that for the system (6) a Choptuik effect will be present only when the scalar fields are coupled non-minimally, i.e. when $\beta \neq const$.

Our investigation of chiral fermions was inspired by the observation of

⁴We are grateful to K. Kuchař for pointing this out to us.

⁵In the context of 2+1 dimensional gravity people speak of black holes also for globally smooth constant curvature solutions where a region containing closed timelike curves is protected from the “outside” world by an event horizon [17].

W. Kummer [9] that, when coupled to the KV model of 2d gravity with torsion [10], chiral fermions allow for a general solution (cf. also [18] for an independent, but later work with the same conclusion). Our work generalizes this result to a broad class of models of 2d gravity as well as to the case of several fermion generations with an additional gauge symmetry.

Moreover, we think that our considerations in Sec. 4 also simplify and illuminate the analysis of [9, 18]: As shown in previous work [13], the purely gravitational part I_{gdl} of the action may be reformulated in terms of a Poisson σ -model [19], the action of which has the form

$$L(A_i, X^i) = \int_{\mathcal{M}} A_i \wedge dX^i + \frac{1}{2} \mathcal{P}^{ij}(X(x)) A_i \wedge A_j \quad , \quad (13)$$

where \mathcal{P}^{ij} is determined explicitly through U, V, W , the indices i, j run over three values, and the A_i and X^i are three one-forms and three zero-forms, respectively (further details, also about subsequent statements, will be provided in Sec. 4 below). It turns out that the addition of I_{ferm} to the gravitational action gives rise to a simple modification of (13): We merely have to add a term

$$\int A_i \wedge J^i \quad , \quad (14)$$

where the one-forms J^i contain fermion variables only. Now one of the main ideas in the realm of σ -models is the use of advantageous coordinates on the *target* space of the theory, which in (13) are the X^i . As demonstrated in [19] and more explicitly in [13], it is possible to choose coordinates $\widetilde{X}^i(X)$ such that the tensor $\mathcal{P}^{\widetilde{i}\widetilde{j}}(\widetilde{X})$ takes a simple form, which, in particular, does no more depend on the (new) field variables \widetilde{X}^i . Now, in the presence of an additional term (14), such a procedure will be helpful only if the transformation does not induce too complicated an \widetilde{X} -dependence of the transformed current $\widetilde{J}^i = \left(\partial \widetilde{X}^i / \partial X^j \right) J^j$. So, in the combined system the task is to find target space coordinates such that the two structures \mathcal{P} and J *simultaneously* take a simple form. As we will see, this is most straightforward if the fermions are chiral. For both chiralities we did not succeed to simplify the coupled system sufficiently.

The above discussion generalizes to a multiplet of fermions with the additional presence of a non-abelian gauge symmetry; the full Lagrangian $I_{gdil} + I_{ferm} + I_{YM}$ then again takes the form of Eqs. (13,14), where now \mathcal{P}^{ij} depends also on the YM-coupling $\alpha(\Phi)$ and the indices i, j run over a larger set of values. With some adaptation the above discussion applies also to the (anti-)self-dual sector of $I_{gdil} + I_{scal}$, as discussed in Sec. 5.

The analysis within this paper remains on the purely classical level. Much (although not all) of the interest in two-dimensional models results from their quantum aspects. We do not know, if the classical solutions found in this paper survive quantization. The chiral solutions could be plagued by chiral anomalies and the quantum integrability of the models in Sec. 3 by anomalies of the diffeomorphism constraints. For a somewhat controversial discussion of the latter topic in the case of the CGHS model cf. [20]; for another attempt to quantize the CGHS model see [21], for semiclassical aspects we refer to the original work [1] as well as to [22] (and references therein). For the quantization of the purely gravitational action (1) cf. [23, 19]; quantization of models resulting from specifications of U, V, W in (1) may be found also in the papers [24]. For thermodynamics of 2d black holes we refer the reader to [25] and [26], the latter method being applicable to theories in any dimension $d \geq 2$.

2 Field Equations of the General Model

The variation of I_{scal} and I_{ferm} with respect to its matter content yields the following generalizations of the massless Klein-Gordon and Dirac equation:

$$\square f + (\ln |\beta|)' \partial^\mu \Phi \partial_\mu f = 0 \quad (15)$$

$$\widetilde{\not{D}}\Psi = 0 \quad (16)$$

where $\square = \nabla^\mu \nabla_\mu$, with ∇ the Riemannian covariant derivative, and where prime denotes differentiation with respect to the argument of the respective

function, a convention kept throughout this paper. Furthermore,

$$\widetilde{\mathcal{D}} = \mathbf{e}_a^\mu \sigma^a \widetilde{D}_\mu, \quad \widetilde{D}_\mu = \partial_\mu - \frac{1}{2} \sigma^3 \omega_\mu + i \widetilde{\mathcal{A}}_\mu, \quad \widetilde{\mathcal{A}}_\mu \equiv \mathcal{A}_\mu - i \partial_\mu \ln \sqrt{|\gamma(\Phi)|}.$$

So both equations (15) and (16) are modified by a term resulting from the dilaton dependent coupling. For the fermions, however, the modification has a particularly simple form: One merely has to add an imaginary part to the $U(1)$ connection \mathcal{A} . Moreover, since this imaginary part is an exact form, the modification may equally well be absorbed into a fermion field with a redefined absolute value: $\widetilde{\Psi} := \sqrt{|\gamma(\Phi)|} \Psi$ satisfies the standard Dirac equation in curved spacetime, $\widetilde{\mathcal{D}} \widetilde{\Psi} = 0$. This becomes also obvious from the form of the action (3), since an unwanted derivative from the first part of the action is seen to cancel against the respective hermitean conjugated term.

This last observation establishes also the conformal invariance of the fermionic action. While the action for the scalar field f is conformally invariant with unmodified f (f has conformal weight zero), Ψ transforms with the inverse fourth root of the conformal factor (Ψ has conformal weight minus one half). Still the total action is not invariant under conformal (or Weyl) transformations due to the presence of the gravitational and $U(1)$ part of the action.⁶

Variation of the action I , Eq. (6), with respect to the gauge field \mathcal{A} yields:

$$\nabla_\nu [\alpha(\Phi) \mathcal{F}^{\mu\nu}] = \gamma(\Phi) \overline{\Psi} \mathbf{e}_a^\mu \sigma^a \Psi. \quad (17)$$

Note that if an action contains a gauge field \mathcal{A} , the solutions of the coupled system do *not* contain those of the system without gauge field as a subsector: if $\mathcal{A} \equiv 0$, the above equation implies $\Psi \equiv 0$. It is an additional equation without counterpart in the system without a gauge field.

We now come to the variation of I with respect to the metric $\mathbf{g}_{\mu\nu}$ (or the

⁶The action for gauge fields is conformally invariant only in four spacetime dimensions (and this on the classical level only); the trace of the energy momentum tensor, Eq. (20) below, vanishes merely for $d = 4$.

zweibein \mathbf{e}_μ^a). One finds

$$\left(\nabla_\mu \nabla_\nu - \mathbf{g}_{\mu\nu} \square\right) U(\Phi) - \frac{1}{2} \mathbf{g}_{\mu\nu} \left(V(\Phi)(\nabla\Phi)^2 + W(\Phi)\right) + V(\Phi) \partial_\mu \Phi \partial_\nu \Phi = \mathbf{T}_{\mu\nu}. \quad (18)$$

Here $\mathbf{T}_{\mu\nu}$ is the energy momentum tensor of the matter fields, $\mathbf{T}_{\mu\nu} = \mathbf{T}_{\mu\nu}^{scal} + \mathbf{T}_{\mu\nu}^{ferm} + \mathbf{T}_{\mu\nu}^{U(1)}$, where

$$\mathbf{T}_{\mu\nu}^{scal} = \beta(\Phi) \left[\partial_\mu f \partial_\nu f - \frac{1}{2} \mathbf{g}_{\mu\nu} \partial_\mu f \partial^\mu f \right], \quad (19)$$

$$\mathbf{T}_{\mu\nu}^{U(1)} = \alpha(\Phi) \left[\mathcal{F}_{\mu\rho} \mathcal{F}_\nu{}^\rho - \frac{1}{4} \mathbf{g}_{\mu\nu} \mathcal{F}_{\lambda\rho} \mathcal{F}^{\lambda\rho} \right]. \quad (20)$$

The energy momentum tensor for the fermion fields is defined by $\mathbf{T}_{\mu\nu}^{ferm} = \mathbf{e}_{a(\mu} \left(\delta I_{ferm} / \delta \mathbf{e}_\nu^a \right) / \det(\mathbf{e}_\rho^b)$, where the brackets around the indices μ and ν indicate symmetrization. For an action that depends on the vielbein only via the combination $\mathbf{g}_{\mu\nu} = \mathbf{e}_\mu^a \mathbf{e}_{a\nu}$ this reduces to the standard expression $\mathbf{T}_{\mu\nu} = 2 (\delta I_{scal} / \delta \mathbf{g}^{\mu\nu}) / \sqrt{-\det \mathbf{g}}$. (In such a case the symmetrization in the indices μ and ν would not be necessary, furthermore).

The fermionic part of the action, Eq. (3), is of the form $\int d^2x \det(\mathbf{e}_\rho^b) \mathbf{e}_\mu^a \mathbf{B}_\mu^a$, resulting in $\mathbf{T}_{\mu\nu} = \mathbf{e}_{a(\mu} \mathbf{B}_{\nu)}^a - \mathbf{g}_{\mu\nu} \mathbf{e}_a^\rho \mathbf{B}_\rho^a$ since \mathbf{B}_μ^a does not depend on the zweibein; this is a particular feature of two spacetime dimensions, where the spin connection ω drops out from the fermionic action altogether. As such $\mathbf{T}_{\mu\nu}$ is not tracefree. However, it is straightforward to show that, as a consequence of the field equations (16) with arbitrary $\gamma(\Phi)$, $\mathbf{e}_a^\rho \mathbf{B}_\rho^a = 0$, so that \mathbf{T}^{ferm} becomes tracefree “on-shell”. (We remark that the symmetrization in the indices μ and ν is essential; the unsymmetrized version remains non-symmetrical also on-shell). The fermionic part of the energy momentum tensor may now be written as

$$\mathbf{T}_{\mu\nu}^{ferm} = \mathbf{e}_{a(\mu} \mathbf{B}_{\nu)}^a, \quad \mathbf{B}_\nu^a \equiv \gamma(\Phi) [\overline{\Psi} \sigma^a (i \overleftrightarrow{\partial}_\nu - \mathcal{A}_\nu) \Psi], \quad (21)$$

where $\overleftrightarrow{\partial}_\mu \equiv (\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)/2$, with $\overleftarrow{\partial}_\mu$ acting on everything to its left except for $\gamma(\Phi)$ outside the brackets and $\overrightarrow{\partial}_\mu = \partial_\mu$.

The field equations (18) may be simplified by taking the trace and elim-

inating $\square U(\Phi)$:

$$\nabla_\mu \partial_\nu U(\Phi) + \frac{1}{2} \mathbf{g}_{\mu\nu} W(\Phi) + V(\Phi) \left(\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} \mathbf{g}_{\mu\nu} \partial_\mu \Phi \partial^\mu \Phi \right) = \mathbf{T}_{\mu\nu} - \mathbf{g}_{\mu\nu} \mathbf{T}^{U(1)}. \quad (22)$$

Here $\mathbf{T}^{U(1)}$ denotes the trace of the $U(1)$ part of the energy momentum tensor, the other two contributions to the trace being zero as a consequence of the conformal invariance of the respective parts of the action.

Finally, variation with respect to the dilaton Φ leads to the equation

$$-U' \mathcal{R} + V' \partial_\mu \Phi \partial^\mu \Phi + 2V \square \Phi - W' + \frac{1}{2} \alpha' \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + \beta' \partial^\mu f \partial_\mu f + 2\gamma' \mathbf{e}_a^\mu \frac{\mathbf{B}_\mu^a}{\gamma} = 0, \quad (23)$$

with \mathbf{B}_μ^a has been defined in Eq. (21).

The action I has several gauge symmetries: it is invariant with respect to diffeomorphisms, local Lorentz transformations in the spin and frame bundle, and $U(1)$ gauge transformations. Despite the corresponding possible simplification of the field equations, it is, to the best of our knowledge, beyond human abilities to solve the general field equations following from I . But even in the restricted cases discussed briefly in the Introduction and to more detail in the following sections, we will not attempt to solve the field equations directly. Instead we will make use of the covariance of a Lagrange formulation with respect to a change of “generalized” coordinates. In the context of field theories generically local transformations of field variables are preferred as usually (and certainly also here) the action is local in the original field variables. Such transformations will be used in all of the following sections to simplify the system as much as possible already at the level of the Lagrangian.

3 General Solution for a Restricted Class of Lagrangians

In this section we discuss the solutions to the Lagrangian I , Eq. (6), with $\beta \equiv 1$, W subject to the condition (7) for some constants a and b , and α

restricted by a similar relation derived below in Eq. (29).

The matter part of the action (except for the gauge field part) is conformally invariant (cf. also the preceding section). A conformal transformation of the Ricci scalar, on the other hand, produces an additive term with two derivatives on the conformal exponent (cf., e.g., [27]). Therefore it is near at hand to get rid of the kinetic term for Φ in I_{gdil} , Eq. (1), by a Φ -dependent conformal transformation. This was first proposed by H. Verlinde [28] for string inspired dilaton gravity and then generalized to I_{gdil} in [29]. We thus change variables from \mathbf{g} to an “auxiliary” metric g defined by:

$$g := \Omega^2(\Phi) \mathbf{g}, \quad \Omega(u) \equiv \exp\left[-\int^u \frac{V(z)}{2U'(z)} dz\right]. \quad (24)$$

Now the action I_{gdil} as a functional of g and Φ again has the form of Eq. (1), but with a potential V that is identically zero, while U remains unchanged and W is replaced by $W(\Phi)/\Omega^2(\Phi)$. Due to the resulting absence of the V -term we now may use $\phi := U(\Phi)$ instead of Φ to also trivialize the potential U to become the identity map.⁷ Distributing the conformal factor in Eq. (24) equally on \mathbf{e}^+ and \mathbf{e}^- , $e^\pm := \Omega \mathbf{e}^\pm$, the spinors are transformed into $\hat{\psi} = \Omega^{-(1/2)} \Psi$. It is, however, also possible to get rid of $\gamma(\Phi)$ by means of the further redefinition (cf. the discussion following Eq. (16)):

$$\psi := \sqrt{\frac{|\gamma(\Phi)|}{\Omega(\Phi)}} \Psi. \quad (25)$$

In the new variables $g = 2e^+e^-$, ϕ , and ψ , the action I takes the form⁸

$$I[g, \phi, f, \psi, A] = \frac{1}{2} \int d^2x \sqrt{-\det g} \left[-\phi R(g) - \widetilde{W}(\phi) + \widetilde{\beta}(\phi) \partial^\mu f \partial_\mu f + \right.$$

⁷This works locally only (and for a non-constant U certainly, which we will assume); for the purpose of solving the field equations in local patches, this is sufficient — except possibly for additional solutions where Φ takes the *constant* value of an extremum of U and which would have to be analyzed separately. A similar remark may hold for the change of variables in (24) certainly.

⁸Here and in what follows we assume $\gamma > 0$; for $\gamma < 0$, B_μ^a is to be replaced by $-B_\mu^a$ in Eq. (26).

$$+ 2e_a^\mu B_\mu^a + \frac{1}{2}\tilde{\alpha}(\phi)F^{\mu\nu}F_{\mu\nu}], \quad (26)$$

$$\text{with} \quad B_\mu^a \equiv \bar{\psi}\sigma^a(i\overleftrightarrow{\partial}_\mu - \mathcal{A}_\mu)\psi, \quad \widetilde{W}(U(\Phi)) = \frac{W(\Phi)}{\Omega^2(\Phi)}, \quad (27)$$

$$\tilde{\alpha}(U(\Phi)) = \Omega^2(\Phi)\alpha(\Phi), \quad \tilde{\beta}(U(\Phi)) = \beta(\Phi). \quad (28)$$

Indices are raised by means of the “auxiliary metric” g and we made use of $\sqrt{-\det g} = \det e_\mu^a$. Note that (26) is in the same form as (6), just with the appropriate replacement of variables and a specification of “potentials”, resulting from putting some of the general ones into the transformation formulas for variables. Therefore the field equations in the new variables follow from those of the previous section.

The intention of the present section is to study models in which the *auxiliary* metric g is one of constant curvature. Thus we require that $\widetilde{W}(\phi)$ is at most linear in ϕ , while $\tilde{\alpha}$ and $\tilde{\beta}$ have to be independent of this field. The first condition is precisely (7), $\widetilde{W}(\phi) = 4a\phi + 2b$ for two constants a and b , while the second one implies

$$\alpha(\Phi) = \alpha_0 \exp\left[\int^\Phi \frac{V(z)}{U'(z)} dz\right] \quad (29)$$

for some constant α_0 ; β , on the other hand, has to be constant altogether and we will normalize it to $\beta = 1$. Note that, due to our definition of ψ , we are not forced to also put γ' to zero.

In the new variables the metric g decouples completely from the matter sector. Up to a choice of coordinates, $R(g) = -4a$ forces g to take the form

$$g = \frac{2dx^+dx^-}{(1 + ax^+x^-)^2} \equiv 2\exp(2\xi)dx^+dx^- , \quad (30)$$

where the function ξ is defined by this equation. Note that this does not imply that the gravitational and matter sectors of the original model decouple. The situation is quite analogous to a system of coupled harmonic oscillators: There the introduction of appropriate variables (“normal coordinates”) leads to a system of decoupled harmonic oscillators. Also here the field equations in the new variables g etc. take qualitatively the same form as

the one of the original variables \mathbf{g} etc., just that, upon restriction to the class of Lagrangians considered in this section, in the new variables the equations of motion simplify greatly and (in part) decouple. (Certainly, if one counts the dilaton to the gravitational sector, this decoupling is not complete. The equations for the dilaton depend on the energy momentum tensor, cf. Eqs. (22) or Eqs. (41,34) below. It is also in this way that the matter enters the metric \mathbf{g} , cf. Eq. (24) above).

Due to $\beta = \tilde{\beta} = 1$ we find f to be a superposition of left- and right-movers,

$$f = f_+(x^+) + f_-(x^-) , \quad (31)$$

just as in flat Minkowski space. This is the case since due to the conformal invariance of (15) with $\beta' = 0$ and f having conformal weight zero, $\square f = 0$ reduces to just $\partial_+ \partial_- f = 0$ for any metric in the conformal gauge. So here $\mathbf{g} \rightarrow g$ does not make any difference.

No gauge fields ($\alpha_0 = 0$):

Next we turn to the field equations for the redefined fermion fields ψ . For simplicity we restrict ourselves to the case of no $U(1)$ gauge field first ($\alpha_0 = 0$ in Eq. (29)). As a consequence of the reformulation we merely have to solve the massless Dirac equation with the background metric g (even despite the fact that $\gamma' \neq 0$). Here it is not so much decisive that g is a space of constant curvature, the main point is that we know its conformal factor *explicitly*, cf. Eq. (30). In such a case the solution of the Dirac equation is the one of Minkowski space, i.e. again consisting of right- and left-movers, $\chi_R(x^+)$ and $\chi_L(x^-)$, conformally *transformed* to the space with metric g as a field with conformal weight minus one half:

$$\psi = \exp(-\xi/2) \begin{pmatrix} \chi_R(x^+) \\ \chi_L(x^-) \end{pmatrix} , \quad (32)$$

where, in the case under consideration, ξ is given by Eq. (30). (Note that the result is no more a superposition of a left- and a right-mover, ξ depending on x^+ and x^- , but only a conformal transform thereof).

Up to now the solution of the field equations was immediate. Now we come to solving the equations (22), however, which is a less trivial task (for $a \neq 0$). Due to the introduction of g and ϕ as new variables, there is no V -term and U is linear. The essential restriction of this section, Eq. (7), moreover, ensures that the whole system becomes linear in ϕ . So we are left with the following three equations

$$\nabla_{\pm} \nabla_{\pm} \phi = \mathbf{T}_{\pm\pm} \quad (33)$$

$$e^{-2\xi} \partial_+ \partial_- \phi + 2a\phi = -b \quad (34)$$

where $\xi_{,\pm} \equiv \partial_{\pm} \xi$ and where we used $\mathbf{T}_{+-} = 0$ (as a consequence of $\mathcal{A} = 0$). The lefthand side of the first two equations may be rewritten more explicitly as: $(\partial_{\pm} - 2\xi_{,\pm}) \partial_{\pm} \phi \equiv e^{2\xi} \partial_{\pm} e^{-2\xi} \partial_{\pm} \phi$. (The only non-vanishing components of the Christoffel connection are $\Gamma^+_{++} = 2\xi_{,+}$ and $\Gamma^-_{--} = 2\xi_{,-}$). The respective righthand sides of Eqs. (41) are given by

$$\mathbf{T}_{++} = \mathbf{T}_{++}(x^+) \equiv (f'_+)^2 + i\chi_R^* \overleftrightarrow{\partial}_+ \chi_R, \quad (35)$$

$$\mathbf{T}_{--} = \mathbf{T}_{--}(x^-) \equiv (f'_-)^2 + i\chi_L^* \overleftrightarrow{\partial}_- \chi_L. \quad (36)$$

Here the indices $+$ and $-$ are world sheet indices (μ, ν, \dots), not to be mixed up with the frame bundle indices (a, b, \dots).

Note that with both indices lowered (and only then!) the energy momentum tensor $\mathbf{T}_{\mu\nu}$ is invariant with respect to the conformal field redefinition (24): $\mathbf{T}_{\mu\nu} = T_{\mu\nu}$. Here the bold faced quantity is the energy momentum tensor of the *original* theory (6), given in Eqs. (19,20,21), and $T_{\mu\nu}$ the energy momentum tensor following from (26) upon variation with respect to the auxilary metric g . This holds also for the $U(1)$ gauge field; the fact that it is not conformally invariant in two dimensions, in contrast to scalar fields, e.g., is reflected in that the redefinition of α to $\tilde{\alpha}$ contains the conformal factor, while the latter is absent in the analogous transition $\beta \rightarrow \tilde{\beta}$ (cf. Eq. (28)).

As $\mathbf{T}_{\pm\pm} = \mathbf{T}_{\pm\pm}(x^{\pm})$ the solution of (41,34) is immediate for $a = 0$, since then also $\xi \equiv 0$ (cf. Eq. (30)); the complication arises when allowing for $a \neq 0$ in Eq. (7).

The general local solution to the equations (41, 34) is of the form:

$$\phi = \mathcal{T}_+(x^+) + 2\xi_{,+} \int^{x^+} \mathcal{T}_+(z) dz + (+ \leftrightarrow -) - \frac{bx^+x^-}{1+ax^+x^-} \quad (37)$$

with

$$\mathcal{T}_\pm(u) = \int^u dv \int^v \mathbf{T}_{\pm\pm}(z) dz + \frac{1}{2}K + k_\pm x^\pm, \quad (38)$$

where in the last line we displayed constants of integration K and k_\pm explicitly. There was no need to introduce two different constants of integration instead of K as within (37) they anyway would contribute the same (i.e. their difference drops out and without loss of generality the two constants may be set equal — assuming that the lower boundaries in the integrations are fixed in some arbitrary way). The constants of integration C_\pm from the integral over \mathcal{T}_\pm , on the other hand, may be absorbed into a redefinition of k_\pm for $a \neq 0$, $k_\pm \rightarrow k_\pm + C_\pm/a$, while they disappear for $a = 0$ due to $\xi_{,\pm} = 0$; so we did not display them.

That Eq. (37) is a solution of the coupled system (41, 34) may be established readily using two relations, following from the definition of ξ in Eq. (30): $\partial_\pm \partial_\pm \xi = (\xi_{,\pm})^2$ and $\partial_+ \partial_- \xi = -a \exp(2\xi)$, the latter of which is equivalent to the statement that $g = 2 \exp(2\xi) dx^+ dx^-$ describes a space of constant curvature $R(g) = -4a$. These two equations are *essential* for (37) to solve Eqs. (41, 34). Thus despite the simplicity and the apparent generality of our solution (37), it solves the field equations only for the specific function ξ defined above! Having found a particular integral of the linear equations (41, 34), one is left with finding the general solution of the homogenous system. As we will see right below, the homogenous solutions ϕ_{hom} are indeed incorporated in (37) taking into account the freedom in choosing the constants K and k_\pm . This then concludes the proof that (37) is the general solution of Eqs. (41, 34).

To our mind the homogenous system is most transparent in a different coordinate system, namely the one in which $g = 2dx^0 dx^1 + 2a(x^0 dx^1)^2$, resulting from (30) by means of $x^- = x^1$ and $x^+ = x^0/(1 - ax^0 x^1)$. These coordinates have the advantage that the zero-zero component of the homogenous system

$\nabla_\mu \partial_\nu \phi_{\text{hom}} + g_{\mu\nu} (2a\phi_{\text{hom}} + b) = 0$ reduces to $\partial_0 \partial_0 \phi_{\text{hom}} = 0$, so that ϕ_{hom} is found to be at most linear in x^0 ! The x^1 dependence is then restricted by the remaining two equations, which are $(\partial_1 - 2ax^0)\partial_0 \phi_{\text{hom}} + 2a\phi_{\text{hom}} = 0$ and $(1 - x^0 \partial_0)\partial_1 \phi_{\text{hom}} = 0$, where we have made use of the former equation to simplify the one-one component of the field equations to reduce to the latter equation. This then leads to

$$\begin{aligned} \phi_{\text{hom}} &= k_+ x^0 + k_- x^1 (1 - ax^0 x^1) + K(1 - 2ax^0 x^1) \\ &= \frac{k_+ x^+ + k_- x^- + K(1 - ax^+ x^-)}{1 + ax^+ x^-} \end{aligned} \quad (39)$$

for three free constants k_\pm and K . As the notation already suggests, they indeed coincide with the three constants found in (37).

As remarked already in Sec. 1, in the absence of matter fields and with a spacetime-topology \mathbb{R}^2 , the general solution of the field equations modulo gauge symmetries is parametrized by *one* real quantity only (cf. [12]). As the above three constants K and k_\pm remain in the matterless (“vacuum”) solution, not all of them can describe physically (or geometrically) different spacetimes. Indeed, Eq. (30) has a residual gauge freedom:

$$x^\pm \rightarrow \lambda^{\pm 1} \frac{x^\pm + s^\pm}{1 - s^\mp a(x^\pm + s^\pm)}, \quad (40)$$

parametrized by three real constants $\lambda \neq 0$ and s^\pm . For $s^\pm := 0$, e.g., this is nothing but the rescaling $x^+ \rightarrow \lambda x^+$, $x^- \rightarrow x^-/\lambda$, leaving (30) invariant at first sight; by means of Eq. (39) it leads to the identification $(k_+, k_-) \sim ((k_+/\lambda), \lambda k_-)$. A more detailed analysis shows that the factor space of K and k_\pm modulo the action induced on them by Eq. (40) is indeed a one parameter space. (E.g., for $a = 0$, $b \neq 0$, and no matter fields, $T_{\mu\nu} = 0$, one easily establishes that k_\pm may be put to zero by shifts in $x^\pm \rightarrow x^\pm + s^\pm$; thereafter rescalings with λ have no effect anymore, so that K remains as a physical parameter. — If, on the other hand $a = b = 0$ and there are no matter fields, then one can achieve, e.g., $k_+ = 1$ and $K = 0$, with k_- remaining.) In many cases the remaining parameter may be given the interpretation of the “mass” M of the spacetime described by \mathbf{g} . Strictly speaking, the above consideration

applies to the matterless case (reconsidered in the present gauge). However, we think that the present *counting* of gauge-invariant parameters will be unmodified when adding matter, so that also in the general case only one of the three parameters k_{\pm} and K will survive.

In summary we get the following results: Combining the solution (37) with Eqs. (24), (30), and (7), the metric is brought into the form of Eq. (11), while $\Phi = U^{-1}(\phi)$. The scalar field is given by Eq. (31), the fermionic field by solving Eqs. (25,32) for Ψ . The local solutions are parametrized by the choice of the one-argument functions f_{\pm} and χ_{\pm} as well as by the one constant of the vacuum theory (cf. the discussion above). In terms of these data the energy momentum tensor, as defined in Eqs. (19,21), has the form (35,36) (while $\mathbf{T}_{+-} = 0$).

Inclusion of gauge fields (arbitrary α_0):

We now turn to the system with an additional $U(1)$ gauge symmetry. Assuming α in the action (5) to be subject to the constraint (29), we obtain a constant coupling $\tilde{\alpha} = \alpha_0$ in the reformulated action (26). The matter sector of the theory still poses no problem, as it is mapped to a corresponding system on a space of constant curvature. So we are left with the dilatonic equations (22), or, equivalently, with:

$$e^{2\xi} \partial_{\pm} e^{-2\xi} \partial_{\pm} \phi = \mathbf{T}_{\pm\pm} \quad (41)$$

$$\partial_+ \partial_- \phi - 2(\partial_+ \partial_- \xi) \phi = -b e^{2\xi} - \mathbf{T}_{+-}. \quad (42)$$

For $\mathbf{T}_{+-} = 0$ this takes the form of Eqs. (41,34). However, the present $\mathbf{T}_{\mu\nu}$ has decisive differences to the previous considerations: First, \mathbf{T}_{++} and \mathbf{T}_{--} depend on *both* coordinates x^+ and x^- now (otherwise we would be forced to $\mathcal{F} = 0$ and, consequently, also to $\Psi = 0$, cf. the remarks following Eq. (17)!). Second, also $\mathbf{T}_{+-} = -(\alpha_0/2) \exp(-2\xi) [\mathcal{F}_{+-}]^2 \neq 0$ for $\mathcal{F} \neq 0$; i.e. due to the presence of the $U(1)$ field the energy momentum tensor is no more tracefree except for $\Psi = 0 = \mathcal{F}$.

The form of $\mathbf{T}_{\mu\nu}$ is still restricted by some decisive relations. These may be expressed most elegantly when using the trace T of the energy momentum

tensor with respect to the auxiliary metric, $T = g^{\mu\nu}T_{\mu\nu} = g^{\mu\nu}\mathbf{T}_{\mu\nu}$. (We remind the reader that in view of Eq. (24) $\mathbf{T} = \Omega^2 T \neq T$ in general). The energy momentum tensor satisfies:

$$\partial_+ \partial_- \sqrt{|T|} = 0 \quad (43)$$

$$\partial_{\pm} T = -2 \exp(-2\xi) \partial_{\mp} \mathbf{T}_{\pm\pm} \quad (44)$$

The first of these relations follows from Eq. (17). It allows to write $T = T^{U(1)} = -\alpha_0 [\exp(-2\xi) \mathcal{F}_{+-}]^2$ in the form $-[F_+(x^+) + F_-(x^-)]^2$, where, again through Eq. (17), the functions F_{\pm} are determined by the fermion fields up to an additive constant. The second relation, Eq. (44), is equivalent to $\nabla^{\mu} \mathbf{T}_{\mu\nu} = 0$, where this equation is understood entirely with respect to the *auxiliary* metric g ; it follows from the equations fulfilled by the matter fields, and, simultaneously, it is the integrability condition of the dilatonic system of differential equations. Note that the second relation allows to express \mathbf{T}_{++} and \mathbf{T}_{--} in terms of T and ξ up to a function of the single variable x^+ and x^- , respectively:

$$\mathbf{T}_{++} = -\frac{1}{2} \int_0^{x^-} e^{(2\xi(x^+, \tilde{x}^-))} (\partial_+ T)(x^+, \tilde{x}^-) d\tilde{x}^- + \mathbf{T}_{++}(x^+, 0) \quad (45)$$

with a similar equation for \mathbf{T}_{--} . The functions $\mathbf{T}_{++}(x^+, 0)$ and $\mathbf{T}_{--}(0, x^-)$ may then be eliminated from the right-hand side of the dilatonic system of linear differential equations by a particular integral of the type (37) found already above. One then is left with finding a particular integral of the remaining system, with inhomogeneities determined solely by $T = -[F_+(x^+) + F_-(x^-)]^2$.

This still turns out to be quite a hard problem. After the dust clears, however, it is possible to put the solution to the system (42) into the following form:

$$\begin{aligned} \phi = & \int_0^{x^+} d\tilde{x}^+ \int_0^{x^-} d\tilde{x}^- \left(\frac{a(x^- - \tilde{x}^-)(1 + a\tilde{x}^+x^-)}{1 + a\tilde{x}^+\tilde{x}^-} - 1 \right) \mathbf{T}_{+-}(\tilde{x}^+, \tilde{x}^-) - \\ & - \frac{bx^+x^-}{1 + ax^+x^-} + \int^{x^+} d\tilde{x}^+ \int^{\tilde{x}^+} du \frac{(1 + a u x^-)^2}{(1 + a \tilde{x}^+ x^-)^2} \mathbf{T}_{++}(u, 0) + \\ & + \int^{x^-} d\tilde{x}^- \int^{\tilde{x}^-} dv \frac{(1 + a x^+ v)^2}{(1 + a x^+ \tilde{x}^-)^2} \mathbf{T}_{--}(0, v) + \phi_{hom} . \end{aligned} \quad (46)$$

Here ϕ_{hom} is the homogenous solution (39), containing the one gauge-invariant parameter of the vacuum theory that is left over when taking into account the residual gauge freedom (40) discussed above.

It is a somewhat cumbersome calculation to show that (46) indeed solves the system (42) and we will not provide any further intermediary steps here so as to not become too technical. Actually, it is even not completely obvious to see how the above ϕ reduces to the previous form (37) for $\mathbf{T}_{+-} = 0$ and $\mathbf{T}_{\pm\pm} = \mathbf{T}_{\pm\pm}(x^\pm)$. We recommend this consistency check as an exercise to the reader.

Up to gauge transformations, the general local solutions are parametrized by the choice of the initial data for the matter fields f and Ψ , as well as by the gauge invariant constant contained in ϕ_{hom} as well as one further constant of integration in the field strength \mathcal{F} .

In this section we analysed 2d gravity-matter models in which the *auxiliary* metric g , defined in Eq. (24), has three Killing vectors (cf. Eq. (30)). Basically this was the defining restriction for the class of models considered in this section. We remark that for a completely general model (6) it is also possible to find *those* solutions, in which g has one Killing field (stationary/homogenous auxiliary spacetimes); for this special *subclass* of solutions of the general model, one can reduce the field equations to one ordinary differential equation of third order. We refer to [11] for details.

4 Chiral Solutions

In this section we want to show that chiral fermions coupled to a general gravity action (1) and to gauge fields is a classically solvable system. To start with, we will, however, discuss the fermion-gravity system with both chiralities, disregarding, furthermore, possible gauge fields in a first step; thus to begin with, our Lagrangian has the form $I = I_{gdil} + I_{ferm}$.

Chiral fermions coupled to generalized dilaton gravity:

As remarked already in the introductory section, I_{gdil} may be formulated equivalently by means of the action (13), with I_{ferm} being given by the expression (14). This comes about as follows (cf. also [13]): As shown in the previous section, appropriate field redefinitions brought I into the form of Eq. (26) with $\tilde{\alpha} = 0 = \tilde{\beta}$. Now we want to reexpress this action for g , ϕ , and ψ in its Cartan formulation, using an (auxiliary) zweibein e^a with $g = 2e^+e^-$ and an (auxiliary) spin connection ω . In this context it is important to note, however, that the zero torsion condition for ω does not result automatically from the variation of the 2d gravity action as it would in the 4d Einstein Hilbert action. Consequently, using ω as an independent variable, we need Lagrange multiplier fields, X^\pm , to enforce zero torsion $De^a \equiv de^a + \varepsilon^a_b \omega \wedge e^b = 0$. The resulting gravitational part of the action,⁹ $\int_{\mathcal{M}} \phi d\omega + X_a De^a + \widetilde{W}(\phi)e^- \wedge e^+ / 2$, may be rewritten identically as given in Eq. (13), if we collect fields according to $A_i \equiv (e^+, e^-, \omega)$ and $X^i \equiv (X^-, X^+, X^3)$ and define the matrix \mathcal{P}^{ij} , $i, j \in \{-, +, 3\}$, by

$$(\mathcal{P}^{ij}) = \begin{pmatrix} 0 & -\frac{1}{2}\widetilde{W}(X^3) & -X^- \\ \frac{1}{2}\widetilde{W}(X^3) & 0 & X^+ \\ X^- & -X^+ & 0 \end{pmatrix}. \quad (47)$$

After application of the 2d identity $\det(e^b_\rho) e^\mu_a = \varepsilon(ab)\varepsilon(\mu\nu) e^b_\nu$, where $\varepsilon(\cdot\cdot)$ denotes the 2d antisymmetric symbol (without any metric dependence), the fermion part of the action, on the other hand, takes the form (14) with $J^j_\mu = (B^-_\mu, -B^+_\mu, 0) = (i\psi_R^* \overleftrightarrow{\partial}_\mu \psi_R, -i\psi_L^* \overleftrightarrow{\partial}_\mu \psi_L, 0)$, where $\psi_{R,L}$ denote the positive/negative chirality components of ψ . Thus, up to an irrelevant factor, the total action $I = I_{gdil} + I_{ferm}$ becomes

$$I = \int_{\mathcal{M}} A_i \wedge (dX^i + J^i) + \frac{1}{2} \mathcal{P}^{ij} A_i \wedge A_j \quad (48)$$

with $i, j \in \{-, +, 3\}$ and J as given above. As we stay on the purely classical level within this paper, the fermionic variables may be taken commuting and

⁹Here we use conventions $\varepsilon^{+-} = 1$ and $\sqrt{-\det g} d^2x = e^+ \wedge e^-$.

J may be simplified by the following parametrization:

$$\psi \equiv \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} := \begin{pmatrix} r \exp(-i\rho) \\ l \exp(i\lambda) \end{pmatrix} \Rightarrow J^i = (r^2 d\rho, l^2 d\lambda, 0) . \quad (49)$$

Following the remarks in the opening section, a change of variables in the target space of the theory, $X^i \rightarrow \widetilde{X}^i$, inducing a change of variables $A_i \rightarrow \widetilde{A}_i = (\partial X^k / \partial \widetilde{X}^i) A_k$, may be used to simplify the tensor \mathcal{P}^{ij} . In particular, as a consequence of the Jacobi identity

$$\mathcal{P}^{il} \partial_l \mathcal{P}^{jk} + \text{cyclic}(ijk) = 0 \quad (50)$$

fulfilled by \mathcal{P}^{ij} as given in Eq. (47) above, there always exist coordinates \widetilde{X}^i such that $\widetilde{\mathcal{P}}^{ij}$ takes (constant) Casimir Darboux (CD) form and the action becomes

$$I = \int_{\mathcal{M}} \widetilde{A}_i \wedge (d\widetilde{X}^i + \widetilde{J}^i) + \widetilde{A}_2 \wedge \widetilde{A}_3 \quad (51)$$

with $i \in \{1, 2, 3\}$.

Note, however, that now the originally simple currents (49) may have become complicated in the transition $J^i \rightarrow \widetilde{J}^i \equiv (\partial \widetilde{X}^i / \partial X^a) J^a$.

A possible choice of CD coordinates of (47) is provided by [13]

$$\widetilde{X}^i = (2X^+ X^- - \int^{X^3} \widetilde{W}(t) dt, \ln |X^+|, X^3) \quad . \quad (52)$$

This induces the following form for the current:

$$\widetilde{J}^i = (2X^+ J^- + 2X^- J^+, J^+ / X^+, 0) , \quad (53)$$

where X^\pm are to be understood as functions of the new variables \widetilde{X}^i . While in the case of pure gravity, $J^i \equiv 0$, also the last of the initially three potentials, $\widetilde{W}(\phi)$, could be eliminated from the action by means of the above change of variables, it now creeps in again through \widetilde{J}^i . *However*, from Eq. (53) it is obvious that $\widetilde{W}(X^3)$ can be eliminated in the case of *chiral* fermions: $\psi_L = 0$ implies $J^+ = 0$ and \widetilde{W} drops out of (51)! With the further (Lorentz invariant) field redefinition

$$\widetilde{r} := 2X^+ r^2 \quad (54)$$

the transform of the chiral current $J^i = (J^-, 0, 0)$ is $\tilde{J}^i = (\tilde{r}d\rho, 0, 0)$ and the action (51) takes the trivialized form

$$I_{chiral} = \int_{\mathcal{M}} \tilde{A}_i \wedge d\tilde{X}^i + \tilde{r}\tilde{A}_1 \wedge d\rho + \tilde{A}_2 \wedge \tilde{A}_3 . \quad (55)$$

We are thus left with solving the field equations of this very simple action.

Variation with respect to \tilde{A}_2, \tilde{A}_3 yields

$$\tilde{A}_2 = d\tilde{X}^3, \quad \tilde{A}_3 = -d\tilde{X}^2, \quad (56)$$

which may be used to eliminate these fields together with \tilde{X}^2, \tilde{X}^3 . This simplifies (55) further to

$$I_{chiral} = \int_{\mathcal{M}} \tilde{A}_1 \wedge (d\tilde{X}^1 + \tilde{r}d\rho). \quad (57)$$

According to (57), \tilde{A}_1 is exact. Thus, with an appropriate choice of coordinates, $\tilde{A}_1 = dx^1/2$. This may be used to establish that all the remaining fields ρ, \tilde{r} , and \tilde{X}^1 are functions of x^1 only and the latter is determined up to a constant of integration c by the former two via

$$\tilde{X}^1 = \tilde{X}^1(x^1) = - \int \tilde{r}(x^1)\rho'(x^1)dx^1 + c. \quad (58)$$

The gauge freedom of our gravity theory may be fixed completely by choosing \tilde{X}^3 as the second coordinate x^0 and by using a Lorentz frame such that $X^+ = 1 \Rightarrow \tilde{X}^2 = 0$.

Thus in the present framework the field equations could be trivialized. The local solutions are parametrized by the choice of initial data for $\tilde{r}(x^1)$, $\rho(x^1)$, and the integration constant c , furthermore. The latter is the only parameter remaining in the matterless case, in coincidence with the literature [12].

We finally reexpress the solution in terms of our original variables \mathbf{g}, Φ , and Ψ , i.e. we perform the transformation inverse to the one that has led us from the two parts (1) and (3) of the original action I (with the additional constraint $\Psi_L = 0 \Leftrightarrow \psi_L = 0$) to the action (55) or (57). In the gauge chosen

above the dilaton has the form $\Phi = U^{-1}(x^0)$. The chiral fermion field is given by (cf. Eq. (25))

$$\Psi_R = \sqrt{\frac{\tilde{\Omega}(x^0)}{|\gamma(U^{-1}(x^0))|}} \psi_R(x^1) , \quad \psi_R(x^1) \equiv r(x^1) \exp[-i\rho(x^1)] , \quad (59)$$

where $\tilde{\Omega} \equiv \Omega \circ U^{-1}$. By means of

$$\begin{aligned} A_- &= 2\tilde{A}_1 = dx^1 , \\ A_+ &= \exp(-\tilde{X}^2) \left[\tilde{A}_2 + \left(\tilde{X}^1 + \int^{\tilde{X}^3} \tilde{W}(t) dt \right) \tilde{A}_1 \right] = \\ &= dx^0 + \frac{1}{2} \left(\tilde{X}^1 + \int^{x^0} \tilde{W}(t) dt \right) dx^1 , \end{aligned} \quad (60)$$

furthermore, the physical metric $\mathbf{g} = g/\tilde{\Omega}^2(x^0) = 2A_+A_-/\tilde{\Omega}^2(x^0)$ (cf. Eq. (24)) becomes

$$\mathbf{g} = \frac{1}{\tilde{\Omega}^2(x^0)} \left[2dx^0dx^1 + \left(-2 \int^{x^1} \mathbf{T}_{11}(u) du + c + \int^{x^0} \tilde{W}(t) dt \right) (dx^1)^2 \right] . \quad (61)$$

Here we rewrote Eq. (58) by means of $\mathbf{T}_{11} = r^2(x^1)\rho'(x^1)$, which follows directly from Eqs. (21,24,25). We remark that \mathbf{T}_{11} is the *only* nonvanishing component of $\mathbf{T}_{\mu\nu}$ here, and that it has support along null lines ($x^1 = \text{const}$ is null according to Eq. (61)).

A further coordinate change $x^0 \rightarrow \int^{x^0} dt/\tilde{\Omega}^2(t)$ brings the metric (61) into the form (9), announced in Sec. 1; the functions h , k_0 , and h_1 may be easily identified from the above.

Chiral fermions coupled to gravitational actions with non-zero torsion:

We briefly discuss changes that occur, if one allows for gravitational actions with torsion terms. Such actions result [13], if one allows the potential \tilde{W} to depend also on X^+X^- in addition to X^3 in Eq. (47). E.g., the Katanaev–Volovich model [10] results from $\tilde{W}(2X^+X^-, X^3) = -\alpha X^+X^- - (X^3)^2 + \Lambda/\alpha^2$ for two real constants α and Λ . In this case g is already the true (physical) metric, without any additional conformal transformation.

Now CD coordinates have the form

$$\widetilde{X}^i = (C(2X^+X^-, X^3), \ln|X^+|, X^3), \quad (62)$$

where the two-argument function $C(u, v)$ is a solution to the differential equation $\widetilde{W}(u, v)\partial_u C + \partial_v C = 0$. For at most linear (X^+X^-) -dependence of \widetilde{W} (such as is the case for the KV-model), C may be determined explicitly from this differential equation (cf. [13]). However, to determine the general solution of the field equations, this is not necessary; it may be written in terms of the function C . Actually, using (62), *all* the steps and formulas from (51) to (58) apply also in this case, except for (53), which generalizes to $\widetilde{J}^1 = 2(\partial_u C) [\exp(\widetilde{X}^2)J^- + X^-(\widetilde{X})J^+] / 2$, where the second term drops out upon restriction to chiral solutions $J^+ \equiv 0$. The Casimir function C enters only when determining the metric from the fields \widetilde{A}_i . E.g., g now becomes

$$g = 2\partial_u C [dx^0 dx^1 + X^- \partial_u C (dx^1)^2], \quad (63)$$

where $\partial_u C$ and X^- are to be understood as functions of the CD-coordinates (62), which, as before, are given by Eq. (58) and $\widetilde{X}^2 = 0$, $\widetilde{X}^3 = x^0$.

In the particular case of the KV-model the above formulas reproduce the results found in [9, 18]. Note that if instead of $J^+ \equiv 0$ we put $J^- \equiv 0$, one still may proceed as above, merely replacing $\widetilde{X}^2 = \ln|X^+|$ by $\widetilde{X}^2 = -\ln|X^-|$.

In [18] it has been claimed that the general solution for fermions of *both* chiralities may be obtained “in the same manner” as those of one chirality. We did not succeed to verify this (cf. also the remarks around Eq. (53)). It would be very interesting to see this general solution of the KV-model coupled to fermions of both chiralities. In particular, from the present perspective it seems most likely that such a solvability would generalize to the *whole class* of models (1) with the same matter content.

Chiral fermions coupled to generalized dilaton gravity and a $U(1)$ gauge field:

We now turn to the discussion of additional gauge fields. We start with $U(1)$ gauge fields, taking the gravitational part torsion free for simplicity,

$I = I_{gdl} + I_{ferm} + I_{U(1)}$, Eqs. (1,3,5). In [13] it has been shown that a gravity Yang–Mills system of the above kind (but without the matter contribution I_{ferm}) may be described by a Poisson σ –model (13) where the indices i run from one to $d + 3$, d being the dimension of the structure group G of the Yang–Mills theory. In the present case of the abelian group $G = U(1)$ we have $d = 1$ and the Poisson tensor \mathcal{P}^{ij} has the same form as (47) with the addition of a forth row and line with zeros and the replacement $\widetilde{W}(X^3)$ by $\widehat{W}(X^3, X^4) = \widetilde{W}(X^3) - (X^4)^2/\tilde{\alpha}(X^3)$. This form of the total gravity– $U(1)$ –action comes about when bringing the $U(1)$ –action, Eq. (5) or better the last term of Eq. (26), into first order form:

$$I_{U(1)} = \int \mathcal{A} \wedge dE + \frac{1}{2\tilde{\alpha}} E^2 e^+ \wedge e^-; \quad (64)$$

this yields the previous form of the $U(1)$ –action upon elimination of the “electric field” E by means of its equations of motion. (The latter is $E = -\tilde{\alpha} * dA$, if $*$ denotes the Hodge dual with respect to g , or, expressed in the original variables, $E = -\alpha * dA$, where $*$ denotes the Hodge dual with respect to \mathbf{g} , α and $\tilde{\alpha}$ being related through Eq. (28)). The gravity– $U(1)$ –action then takes Poisson σ –form with the Poisson tensor as described above and the identifications $A_4 = \mathcal{A}$ and $X^4 = E$ in addition to those for A_i and X^i , $i \in \{-, +, 3\}$, made in the absence of the gauge field. Alternatively, A_4 may be defined also as in Eq. (66) below. This will turn out to be more convenient in the case of chiral fermions, while it does not change the form of \mathcal{P}^{ij} . For the moment, however, we will stick to $A_4 = \mathcal{A}$, as in the matterless case discussed in [13].

As we saw already in Sec. 2, in contrast to the spin connection, the $U(1)$ connection \mathcal{A} does not drop out from I_{ferm} (cf., e.g., Eq. (21)). The additional contribution in I_{ferm} containing the connection one–form is $\int \gamma(\Phi) (-\Psi_R^* \Psi_R e^+ + \Psi_L^* \Psi_L e^-) \wedge \mathcal{A} = \int (-\psi_R^* \psi_R A_- + \psi_L^* \psi_L A_+) \wedge A_4$ (using $A_4 = \mathcal{A}$). Although this is of the form $A_i \wedge A_j$ with an appropriate coefficient matrix, it is *not* advisable to incorporate it in \mathcal{P}^{ij} . The reason is that the thus redefined tensor \mathcal{P} would no more satisfy the Jacobi identity (50)! Since the latter is at

the heart of our approach, we proceed differently: With the currents

$$J^i = (r^2 (d\rho - \mathcal{A}), l^2 (d\lambda + \mathcal{A}), 0, 0), \quad (65)$$

where we used the parametrization (49) again, the coupled system takes the form (48), with i, j running over four values now.

The current (65) suggests to change variables from \mathcal{A} to the gauge invariant combination $\mathcal{A} - d\rho$. We thus (re)define A_4 by:

$$A_4 := \mathcal{A} - d\rho. \quad (66)$$

In the case of one chirality, $\Psi_L := 0 \Leftrightarrow l = 0$, to which we will restrict ourselves in the following, the only non-vanishing component of the current becomes $J^- = -r^2 A_4$ then, while the action again takes the form (48) (with the same four times four matrix \mathcal{P} as above). The field ρ is now seen to drop out completely from the action as a total divergence (exact two-form). This is a manifestation of the $U(1)$ gauge invariance. Dropping the total divergence and implicitly the phase ρ of the fermion field, eliminates the gauge freedom and, at the same time, saves us the study of the associated redundant field equations (according to Noether's second theorem any local symmetry gives rise to a relation among the equations of motion, cf., e.g., [30]).

Next we transform the action to CD coordinates. The extended Poisson tensor has CD coordinates of the form (52) with two changes: First, $\widetilde{W}(X^3)$ is replaced by $\widehat{W}(X^3, X^4) \equiv \widetilde{W}(X^3) - (X^4)^2/\tilde{\alpha}(X^3)$, X^3 being the variable to integrate over, and second there is a fourth coordinate, $\widetilde{X}^4 := X^4$, which now is the second Casimir of the (four by four) matrix \mathcal{P} beside \widetilde{X}^1 . Introducing again the Lorentz invariant function (54) and using Eq. (56) to get rid of some irrelevant field equations¹⁰, we end up with

$$\tilde{I} = \int_{\mathcal{M}} \tilde{A}_1 \wedge d\widetilde{X}^1 + \tilde{A}_4 \wedge (d\widetilde{X}^4 + \tilde{r}\tilde{A}_1) \quad , \quad (67)$$

¹⁰Here again one drops a total divergence, getting rid of the local Lorentz symmetry and half of the diffeomorphism invariance. Certainly the resulting action, Eq. (67) below or Eq. (57) above, respectively, are still invariant under the full diffeomorphism group (as is obvious from its formulation in terms of forms without using any background metric); the

where \tilde{r} is defined as without gauge field. The field equations of the above action are found and solved readily. Thereafter one transforms back to the original variables. We display the result of these considerations only. In an appropriate gauge one finds $\Phi = U^{-1}(x^0)$, $r = r(x^1)$, and $\rho = \rho(x^1)$, as before. \mathcal{A} has the form:

$$\mathcal{A} = -E(x^1) \left(\int^{x^0} \frac{dz}{\tilde{\alpha}(z)} \right) dx^1, \quad (68)$$

where $E = E(x^1) = -\int^{x^1} r^2(z) dz + \hat{c}$. The metric, finally, is given by

$$\begin{aligned} \mathbf{g} = \frac{1}{\tilde{\Omega}^2(x^0)} & \left[2dx^0 dx^1 + \left(\int^{x^0} \tilde{W}(z) dz - 2 \int^{x^1} r^2(t) \rho'(t) dt + c - \right. \right. \\ & \left. \left. - E^2(x^1) \int^{x^0} \frac{dt}{\tilde{\alpha}(t)} \right) (dx^1)^2 \right]. \end{aligned} \quad (69)$$

We recall that $\tilde{\Omega} \equiv \Omega \circ U^{-1}$ and $\tilde{\alpha} = \tilde{\Omega}^2 \cdot (\alpha \circ U^{-1})$, where the function Ω is given by Eq. (24).

The generalization to non-trivial torsion is straightforward.

Multiplet of chiral fermions with YM-fields and generalized dilaton gravity:

The above results may be generalized also to gauge fields of an arbitrary non-abelian structure group G with a chiral fermion multiplet Ψ_R in the fundamental representation of G . G is assumed to allow for a non-degenerate ad-invariant inner product on its Lie algebra, which we will denote by tr . The kinetic term for the Lie algebra valued gauge field \mathcal{A} is of the form $I_{YM} = -\int_{\mathcal{M}} \alpha(\Phi) tr(\mathcal{F} \wedge *\mathcal{F})/2$, where $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the standard field strength and again we may allow for a dilaton dependent coupling α . As shown in detail in [13], $I_{YM} + I_{g\tilde{d}il}$ may be brought into the form (13), with a $(\dim G + 3)$ -dimensional target space. Again all of the matter part of apparent paradox is resolved by noting that the solution for the remaining fields depend on one coordinate function only (x^1 with our choice of coordinates), so that the part of the diffeomorphism group eliminated by means of (56), which is $x^0 \rightarrow x^0(\bar{x}^\mu)$, acts trivially on these solutions.

the action may be formulated as $\int A_i \wedge J^i = \int A_- \wedge J^-$, where now

$$J^- = \gamma(\Phi) \left(\Psi_R^\dagger (i \overleftrightarrow{d} - \mathcal{A}) \Psi_R \right) \quad (70)$$

with $2 \overleftrightarrow{d} \equiv \overrightarrow{d} - \overleftarrow{d}$. So the total action $I_{gdl} + I_{YM} + I_{ferm}$ is again of the form (48) (even in the non-chiral case, but then also with a non-vanishing component J^+ containing the left-handed fermion multiplet).

As CD-coordinates (on the target space) we may choose: $\widetilde{X}^1 = 2X^+X^- - \int^{X^3} \widetilde{W}(t)dt + \text{tr}(E^2) \int^{X^3} (1/\tilde{\alpha}(t)) dt$, where $\tilde{\alpha}$ is related to α via Eq. (28), $\widetilde{X}^2 = \ln|X^+|$, $\widetilde{X}^3 = X^3$, while the following CD-coordinates depend on the (Lie algebra valued) electric fields E only — we do not need to choose the latter functions explicitly to solve the coupled system. In the fermion-YM-sector of the theory we found it most advisable to switch between CD-adapted fields and the original matrix valued fields \mathcal{A} and E .

We do not want to go into the calculational details here. Still we warn of a possible pitfall in using the present formalism: In view of (48) when written with CD-adapted fields, one is tempted to conclude $d\widetilde{X}^1 + \widetilde{J}^1 = 0$; this equation is *wrong*, however! The reason for this failure is the following. Expressing $\widetilde{J}^1 = 2X^+J^-$ in terms of CD-adapted fields, from \mathcal{A} one picks up a contribution proportional to \widetilde{A}_1 : $\widetilde{J}^1 = \widehat{J} + 4X^+\gamma(\Psi_R^\dagger E \Psi_R) \int^{X^3} \frac{dt}{\tilde{\alpha}(t)} \widetilde{A}_1$. This contribution cancels from the action: $\int \widetilde{A}_1 \wedge (d\widetilde{X}^1 + \widetilde{J}^1) + \dots = \int \widetilde{A}_1 \wedge (d\widetilde{X}^1 + \widehat{J}) + \dots$. The correct field equation stemming from the variation with respect to \widetilde{A}_1 is $d\widetilde{X}^1 + \widehat{J} = 0$. As may be seen, $d\widetilde{X}^1 + \widetilde{J}^1 = 0$ would imply $\mathcal{F} = 0$; this is already wrong in the abelian case, discussed in detail before.

At the end of the day one finds the following solution as a straightforward generalization of the abelian results: The metric \mathbf{g} may be brought into the form of Eq. (69), where E^2 generalizes to $\text{tr}E^2$ and $r^2\rho'$ to $i\psi_R^\dagger \overleftrightarrow{\partial} \psi_R$. Here ψ_R and Ψ_R are related by Eq. (25), where again $\psi_R = \psi_R(x^1)$, in an appropriately chosen gauge, and $\Phi = U^{-1}(x^0)$. E is given by $E = E(x^1) = -\int^{x^1} (\psi_R^\dagger(u) T^I \psi_R(u)) du T^I + \widehat{c}$, where T^I , $I = 1, \dots, \dim G$, denote the generators of the Lie algebra and \widehat{c} is some constant of integration restricted to the Cartan subalgebra. \mathcal{A} takes the form of Eq. (68), reinterpreted as

a Lie algebra valued equation. The local solutions are parametrized by the functions $\psi_R(x^1)$ and the $r + 1$ “vacuum parameters” c and \hat{c} , r being the rank of the Lie algebra.

5 Self–dual Scalar Fields

In this section we study the (anti–)self–dual sector of the system Eqs. (1, 2), or, equivalently, of the first line of Eq. (26). While the first two terms of the latter equation may be described by (13), in the following we will bring also the action of the scalar field into a compatible first order form. This will finally allow us to map the (anti–)self–dual sector of the present system to the chiral Lagrangian of the previous section. In this process we will pick up an additional constraint, however. The latter will enforce $\beta = \text{const}$ so as to allow for non–trivial solutions. With an appropriate conversion of symbols, these non–trivial solutions may then be *copied* from the solutions found in the previous section, without the need of solving any field equation. At the end it will turn out among others that the metric again may be brought into the form (61) where now \mathbf{T}_{11} denotes the only non–vanishing component of the energy momentum tensor of the self–dual scalar fields.

We first bring the action for a scalar massless field into first order form¹¹:

$$I_{scal} = -\frac{1}{2} \int_{\mathcal{M}} \beta(\Phi) df \wedge *df \cong \int_{\mathcal{M}} B \wedge df - \frac{1}{2\beta(\Phi)} B \wedge *B, \quad (71)$$

where we introduced a one–form B that equals $\beta(\Phi)*df$ on shell. ‘ $*$ ’ denotes the Hodge dual operation with respect to the dynamical metric \mathbf{g} . However, due to the conformal invariance of the action, and the fact that f carries conformal weight zero, we may equally well take g instead, defined in Eq. (24). Next we split B into its self–dual and its anti self–dual part. Due to $*A_{\mp} = \pm A_{\mp}$, which is equivalent to $*\mathbf{e}^{\pm} = \pm *\mathbf{e}^{\pm}$, this splitting is achieved

¹¹The minus sign in front of the first integral is a consequence of $\varepsilon = -\sqrt{-\det g} d^2x$, following from the conventions chosen in this paper (first footnote of Sec. 4).

by means of the decomposition:

$$B = RA_- + LA_+ , \quad (72)$$

where the notation R and L stands for “right-moving” and “left-moving”, respectively; in particular, the above R has nothing to do with the curvature scalar of g or \mathbf{g} . Combining Eqs. (72) and (71) and using $\beta(\Phi) = \tilde{\beta}(\phi)$, Eq. (28), we obtain

$$\int_{\mathcal{M}} A_- \wedge Rdf + A_+ \wedge Ldf + \frac{RL}{\tilde{\beta}(X^3)} A_- \wedge A_+ \quad (73)$$

to be added to (13). Comparing this with Eqs. (48,49), we see that the first two terms in (73) give rise to a current J^i which is precisely of the form (49). Here this current is even simpler: the scalar field f plays the role of the two phases ρ and λ , which are *equal* (while r^2 corresponds to R and l^2 to L). The third and remaining term in (73), on the other hand, implies a complication: it mixes R - and L -fields and in this respect resembles a (dilaton dependent) mass term for the fermions.

In principle one can absorb this last term into the Poisson structure; it still satisfies the Jacobi identity (50) for $i = 1, 2, 3$, with R and L entering \mathcal{P} as parameters then. However, when changing to CD-coordinates of this Poisson tensor, fields such as \tilde{A}_i depend implicitly on R and L also and thus may not be varied independently of the latter. We therefore will not follow this route.

Inspired by the preceding section, we want to look for chiral solutions, i.e. for solutions satisfying

$$df = *df \quad \Leftrightarrow \quad L = 0 , \quad (74)$$

where the Hodge dual is taken with respect to \mathbf{g} (or g , this makes no difference for one-forms in two dimensions). Similarly we could proceed with $df = -*df \Leftrightarrow R = 0$. In the presence of a (Dirac) mass term, there are no chiral fermion solutions. Similarly here for $\beta'(\Phi) \neq 0$ there are no solutions with $L \equiv 0$ or $R \equiv 0$. However, for a minimal coupling, $\beta = \text{const} \Leftrightarrow \tilde{\beta} = \text{const}$, non-trivial (anti-)self-dual scalar field solutions do exist.

In the preceding section we could implement a condition of the type (74) directly into the Lagrangian. This is no more possible here. Still, if we keep the variation with respect to L , i.e.

$$\left(\tilde{\beta}(X^3)df - RA_-\right) \wedge A_+, \quad (75)$$

as an extra condition, we may thereafter put L to zero in the Lagrangian. But then, up to the renaming $r^2 \rightarrow R$ and $\rho \rightarrow f$, the resulting action is *identical* to the chiral fermion action of the preceding section (i.e. to Eqs. (48, 49)) with $l \equiv 0$). Thus, with the above renaming, we may now just take the solution of the previous section, without the need of solving any field equation! This solution, however, has to satisfy the additional constraint (75) now. Using (60) and taking into account that R and f are functions of x^1 only, Eq. (75) becomes

$$R(x^1) = \tilde{\beta}(x^0) f'(x^1). \quad (76)$$

Obviously this condition can be satisfied only for a constant function $\tilde{\beta}$, proving our previous claim. In view of the qualitative changes in the Klein-Gordon equation induced by a dilaton dependent coupling, cf. Eq. (15), this result is not unpalatable though (note $df = \pm * df \Rightarrow \square f = 0$).

For $\beta = \text{const}$, on the other hand, we find that, in contrast to chiral fermions, the function R may not be chosen freely, but is determined by f . So the local solutions are parametrized by the choice of the “initial data” $f(x^1)$ and c . Noting, furthermore, that $\mathbf{T}_{11}^{ferm} = r^2 \rho'$ translates into $Rf' = \beta(f')^2$, which is nothing but \mathbf{T}_{11}^{scal} (cf. Eq. (19), using $\mathbf{g}^{11} = 0$), we arrive at the following form of the self-dual solutions of $I_{gdil} + I_{scal}$:

$$\Phi = U^{-1}(x^0), \quad f = f(x^1), \quad (77)$$

while the metric \mathbf{g} takes the form (61) with $\mathbf{T}_{11} = \mathbf{T}_{11}(x^1)$ being now the only non-vanishing component of the energy momentum tensor of the scalar field.

Similarly the generalization of the results of the previous section to non-zero torsion are valid here as well.

Concluding we remark that although for $\beta = \text{const}$ the *general* solution of the Klein–Gordon equation (15) consists of a superposition of left– and right–movers, $f = f_+(x^+) + f_-(x^-)$, which are the self–dual parts of f , due to the non–linearities of the coupled system of equations (cf., e.g., Eqs. (22) and (19)), we cannot obtain the general solution from a superposition of the self–dual and anti–self–dual solutions found in the present section. (A similar statement holds for fermions with both chiralities and the chiral solutions found in the preceding section). The simultaneous presence of chiral fermions and self–dual scalar fields, on the other hand, is no problem; this just implies $\mathbf{T}_{11} = \mathbf{T}_{11}^{ferm} + \mathbf{T}_{11}^{scal}$ in Eq. (61).

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